Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE ANALYST.

Vol. II. Nov., 1875. No. 6.

ON THE DEVELOPMENT OF THE PERTURBATIVE FUNCTION IN PERIODIC SERIES.

BY G. W. HILL, NYACK TURNPIKE, N. Y.

1. There are two modes of developing this function. In one, the numerical values of the elements involved are employed from the outset, and the results obtained belong only to the special case treated. This mode has been, almost exclusively, followed by Hansen, and is, perhaps, to be recommended when numerical results are chiefly desired. In the other, all the elements are left indeterminate, and thus is obtained a literal development possessing as much generality as possible. Certain investigations, arising from Jacobi's treatment of dynamical equations and Delaunay's method in the lunar theory, have invested the latter mode of development with additional interest, and with it we shall be exclusively engaged in this article.

In Liouville's Journal for 1860, M. Puiseux has given us two memoirs on this subject, in which appears the general term of this function, but his formulæ seem susceptible of modifications which would render them much simpler. More recently, in the volume of the same journal for 1873, M. Bourget has presented the development in a more concise form by employing the Besselian functions, but as he discards the use of the functions $b_s^{(i)}$, his formulæ, on this account are more complex. It is hoped, that, even if the expressions, given hereafter, are deemed too cumbrous for practical use, they may still possess some interest from a theoretical point of view.

2. It is known that if we have a function S of a variable ζ, which is never infinite, and such that the relation

$$\text{function}\,(\zeta + 2i\pi) = \text{function}\,(\zeta)$$

is satisfied for all integral values of i both positive and negative, it can be developed in a series of the form

$$\Sigma_i\,.\,(K_i^{(c)} \cos i\zeta + \mathrm{K}_i^{(s)} \sin i\zeta),$$

in which i denotes a positive integer; and that, in the cases where this series is infinite, it is convergent.

In general the handling of periodic series is easier if we introduce imaginary exponentials in the place of the circular functions. Thus, ε denoting the base of natural logarithms, we shall put $z = \varepsilon^{\zeta\sqrt{-1}}$, whence

$$2\cos\zeta = z + z^{-1}, \qquad 2\cos i\zeta = z^i + z^{-i},$$
$$2\sqrt{(-1)}\sin\zeta = z - z^{-1}, \qquad 2\sqrt{(-1)}\sin i\zeta = z^i - z^{-i},$$
$$z = \cos\zeta + \sqrt{(-1)}\sin\zeta, \qquad z^i = \cos i\zeta + \sqrt{(-1)}\sin i\zeta.$$

The above theorem then comes to the same thing as to say that S is developable in a series of the form

$$\Sigma_i \,.\, C_i z^i,$$

where the summation is extended to negative as well as positive values of i. The coefficients K are given in terms of the coefficients C by the equations

$$K_i^{(c)} = C_i + C_{-i},$$
$$K_i^{(s)} = (C_i - C_{-i})\sqrt{-1},$$

except the case where $i = 0$, when $K_0^{(c)} = C_0$. It will be seen that when S is real, C_i is a complex number $a + b\sqrt{-1}$, and C_{-i}, its conjugate $a - b\sqrt{-1}$, which renders the coefficients K real, as they should be.

The integral

$$\int z^i d\zeta = \int \Big(\cos i\zeta + \sqrt{(-1)}\sin i\zeta\Big) d\zeta,$$

taken between the limits 0 and 2π, vanishes in all cases except when $i = 0$, when its value is 2π. Hence any function, capable of expansion in a series of positive and negative integral powers of z, integrated with respect to ζ between these limits, gives, as the result, 2π times the coefficient of z^0 in its expansion. And as the coefficient of z^0 in the function Sz^{-i} is evidently C_i, we have

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} Sz^{-i}\, d\zeta.$$

This equation holds for all values of i, negative as well as positive, zero included.

3. Let us now suppose that ζ denotes the mean anomaly of a planet, and let u be the eccentric anomaly, connected with the former by the equation, e being the eccentricity,

$$u - e\sin u = \zeta.$$

In like manner as for ζ, we will introduce the imaginary exponential $s = \varepsilon^{u\sqrt{-1}}$, thus, as the last equation can be written

$$\varepsilon^{(u - e\sin u)\sqrt{-1}} = \varepsilon^{\zeta\sqrt{-1}},$$

by the introduction of the variables s and z, this becomes

$$s\varepsilon^{-\frac{e}{2}\left(s-\frac{1}{s}\right)} = z,$$

which is the transcendental equation connecting s and z. We have

$$d\zeta = (1 - e\cos u)du = \left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]du.$$

Substituting these values in the equation giving the value of C_i, and noticing that, as ζ and u both take the values 0 and 2π together, the limits of integration, when u is the independent variable, are the same as for ζ, we get

$$C_i = \frac{1}{2\pi}\int_0^{2\pi} Ss^{-i}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]du$$

But, from what precedes, we conclude that the coefficient of s^i in the expansion of any function W, according to positive and negative powers of s, is

$$\frac{1}{2\pi}\int_0^{2\pi} Ws^{-i}du.$$

Thus, from the foregoing expression for C_i, we derive the following proposition:—

i *being a positive or negative integer or zero, the coefficient of* z$^{\text{i}}$, *in the development of* S, *according to the powers of* z, *is equal to that of* s$^{\text{i}}$ *in the development of*

$$S\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right],$$

according to the powers of s.

As most of the functions S, which are presented by astronomy for development in powers of z, are quite readily expanded in powers of s, this theorem is of much use. Another form can be given to it. For we have, integrating by parts

$$\int Sz^{-i}d\zeta = -\sqrt{(-1)}\int Sz^{-(i+1)}dz$$

$$= \frac{\sqrt{-1}}{i}Sz^{-i} - \frac{\sqrt{-1}}{i}\int\frac{dS}{dz}z^{-i}dz.$$

Taking the integrals between the limits $\zeta = 0$ and $\zeta = 2\pi$, we get

$$C_i = -\frac{\sqrt{-1}}{2i\pi}\int\frac{dS}{ds}z^{-i}ds$$

$$= \frac{1}{2i\pi}\int_0^{2\pi}\frac{dS}{ds}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}s^{-(i-1)}du.$$

Whence we conclude this proposition: —

The coefficient of z^i *in the development of* S *according to the powers of* z *is equal to that of* s^{i-1} *in the development of*

$$\frac{1}{i}\cdot\frac{dS}{ds}\,\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

according to the powers of s.

This theorem however is not applicable when $i = 0$.

4. We shall often have occasion for the expansion of the function

$$\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

in powers of s; let us, for simplicity, put $\lambda = \frac{ie}{2}$, and

$$\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \Sigma_i\,.\,\underset{\lambda}{J^{(i)}}\,s^i.$$

We have

$$\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}} = \left[1+\frac{\lambda s}{1}+\frac{\lambda^2 s^2}{1.2}+\frac{\lambda^3 s^3}{1.2.3}+\ldots\right]$$
$$\times\left[1-\frac{\lambda}{s}+\frac{1}{1.2}\frac{\lambda^2}{s^2}-\frac{1}{1.2.3}\frac{\lambda^3}{s^3}+\ldots\right],$$

whence we conclude that

$$\underset{\lambda}{J^{(i)}} = \frac{\lambda^i}{1.2\ldots i}\left[1-\frac{\lambda^2}{1.(i+1)}+\frac{\lambda^4}{1.2(i+1)(i+2)}-\ldots\right].$$

This series is not applicable when i is negative; but if, in the function $\varepsilon^{\lambda s}\,.\,\varepsilon^{-\frac{\lambda}{s}}$, we substitute $\frac{1}{s}$ for s, and change the sign of λ, the function remains unchanged, hence

$$\Sigma_i\,.\,\underset{\lambda}{J^{(i)}}\,s^i = \Sigma_i\,.\,\underset{-\lambda}{J^{(i)}}\,s^{-i},$$

and consequently

$$\underset{\lambda}{J^{(-i)}} = \underset{-\lambda}{J^{(i)}} = (-1)^i\underset{\lambda}{J^{(i)}},$$

by which the values of these functions for negative values of i can be derived from those in which i is positive. These functions are known as the Besselian. By putting

$$T_i = 1-\frac{\lambda^2}{1.(i+1)}+\frac{\lambda^4}{1.2(i+1)(i+2)}-\ldots$$

one will have no difficulty in deducing the equation

$$T_{i-1} = T_i - \frac{\lambda^2}{i(i+1)}\,T_{i+1}.$$

5. We come now to the more complex function S of two variables ζ and ζ'; it is known that when this is never infinite and is such that

$$\text{function}\,(\zeta + 2i\pi,\ \zeta' + 2i'\pi) = \text{function}\,(\zeta,\ \zeta')$$

it can be developed in a series of the form

$$\Sigma_{i,\,i'}\,[K^{(c)}_{i,\,i'}\cos(i\zeta + i'\zeta') + K^{(s)}_{i,\,i'}\sin(i\zeta + i'\zeta')],$$

where to one of the quantities i and i', we need assign only positive integral values, but to the other both positive and negative values. If we adopt another imaginary exponential $z' = \varepsilon^{\zeta'\sqrt{-1}}$, this is the same as saying that

$$S = \Sigma_{i,\,i'}\,.\,C_{i,\,i'}z^{i}z'^{i'},$$

where the summation is extended to all integral values positive and negative for i and i'. Since we have

$$\begin{aligned} z^{i}z'^{i'} &= (\cos i\zeta + \sqrt{(-1)}\sin i\zeta)(\cos i'\zeta' + \sqrt{(-1)}\sin i'\zeta') \\ &= \cos(i\zeta + i'\zeta') + \sqrt{(-1)}\sin(i\zeta + i'\zeta'), \end{aligned}$$

the relations, which connect the coefficients K with the coefficients C, are

$$\begin{aligned} K^{(c)}_{i,\,i'} &= C_{i,\,i'} + C_{-i,-i'}\,, \\ K^{(s)}_{i,\,i'} &= (C_{i,\,i'} - C_{-i,-i'})\sqrt{-1}, \end{aligned}$$

unless i and i' are both zero, when

$$K^{(c)}_{0,\,0} = C_{0,\,0}\,.$$

A course of reasoning, similar to that in the case of one variable, establishes that

$$C_{i,\,i'} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} S z^{-i}z'^{-i'}d\zeta\, d\zeta',$$

which holds for all integral values of i and i', positive, negative and zero.

6. Supposing that ζ' denotes the mean anomaly of a second planet, whose eccentricity and eccentric anomaly are respectively e' and u', we have

$$u' - e'\sin u' = \zeta',$$

and by the adoption of the imaginary exponential $s' = \varepsilon^{u'\sqrt{-1}}$, this is transformed into

$$s'\,\varepsilon^{-\frac{e'}{2}\left(s' - \frac{1}{s'}\right)} = z'.$$

It is not difficult to see that we have the following theorem: —

The coefficient of $z^{i}z'^{i'}$ *in the development of* S, *according to the powers of* z *and* z', *is equal to that of* $s^{i}s'^{i'}$ *in the development of*

$$S\varepsilon^{\frac{ie}{2}\left(s - \frac{1}{s}\right)}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]\,.\,\varepsilon^{\frac{i'e'}{2}\left(s' - \frac{1}{s'}\right)}\left[1 - \frac{e'}{2}\left(s' + \frac{1}{s'}\right)\right],$$

according to the powers of s *and* s'.

7. After these preliminaries relative to the general development of functions in periodic series, we come to the matter more immediately engaging our attention. The perturbative function for the action of a planet,

whose mass is m', on another, whose mass is m, is usually written

$$R = m' \left[\frac{1}{\varDelta} - \frac{r \cos \psi}{r'^2}\right],$$

and that for the action of m on m'

$$R_1 = m \left[\frac{1}{\varDelta} - \frac{r' \cos \psi}{r^2}\right],$$

where $\varDelta$ denotes their mutual distance, ψ their angular distance as seen from the sun, and r and r' their radii vectores. The problem proposed is then to develop these two functions in series whose general term is of the form $C_{i,i'} z^i z'^{i'}$. To this end it seems better to discuss the two portions of the general perturbative function, $\frac{1}{\varDelta}$ and $-\frac{r \cos \psi}{r'^2}$, separately, and not, as most investigators, attempt, by a particular notation, to combine, in a whole, these two parts. Thus, in developing $\frac{1}{\varDelta}$, we shall have the term common to both functions, and may suppose that r' denotes the radius vector which belongs to the planet more distant from the sun. But, in treating the second part, we shall suppose that r' belongs to the disturbing planet. The following equations are well known,

$$\begin{aligned}
\varDelta^2 &= r'^2 - 2rr' \cos \psi + r^2, \\
\cos \psi &= \cos (v + \Pi) \cos (v' + \Pi') + \cos I \sin (v + \Pi) \sin (v' + \Pi') \\
&= \cos (v - v' + \Pi - \Pi') - 2 \sin^2 \tfrac{1}{2} I \sin (v + \Pi) \sin (v' + \Pi'),
\end{aligned}$$

where v and v' are the true anomalies, and Π and Π' are the angular distances of the perihelia from either point of intersection of the planes of the orbits, and I is their mutual inclination.

8. Attending then, in the first place, to the development of $\frac{1}{\varDelta}$, we have to notice what are the conditions under which this quantity can be developed in powers of z and z'. In the case of two elliptic orbits, the only one we shall consider here, it is plain that $\frac{1}{\varDelta}$ is always finite and continuous provided the orbits have no point in common. Here we must make two cases according as the value of $\sin I$ is not or is zero. In the first case it is evident that the orbits can meet only on the line of intersection of their planes. Hence, p and p' denoting their semi-parameters, there will be two, one or no points in common, according as two, one or none of the equations,

$$\begin{aligned}
p'(1 + e' \cos \Pi')^{-1} &= p (1 + e \cos \Pi)^{-1}, \\
p'(1 - e' \cos \Pi')^{-1} &= p (1 - e \cos \Pi)^{-1},
\end{aligned}$$

are satisfied. In the second case, where the orbits lie in the same plane, there will be two intersections or none, according as the equation

$$p'[1 + e' \cos(\lambda - \omega')]^{-1} = p[1 + e \cos(\lambda - \omega)]^{-1},$$

λ being the unknown quantity and ω and ω' the longitudes of the perihelia, admits real or imaginary roots. If we put

$$pe' \cos \omega' - p'e \cos \omega = A \cos a,$$
$$pe' \sin \omega' - p'e \sin \omega = A \sin a,$$

this equation takes the form

$$A \cos(\lambda - a) = p' - p.$$

The roots of this are imaginary when

$$(p' - p)^2 > p^2 e'^2 - 2pp'ee' \cos(\omega - \omega') + p'^2 e^2.$$

9. If we put

$$P = r'^2 - 2rr' \cos(v - v' + \Pi - \Pi') + r^2,$$
$$Q = 4 \sin^2 \tfrac{1}{2} I \,.\, r \sin(v + \Pi) \,.\, r' \sin(v' + \Pi'),$$

we have

$$\Delta^2 = P + Q,$$
$$\frac{1}{\Delta} = [P + Q]^{-\frac{1}{2}}$$
$$= P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1}{2} \cdot \frac{3}{4} P^{-\frac{5}{2}} Q^2 - \ldots\ldots,$$

a series we shall denote thus

$$\frac{1}{\Delta} = \sum_{k=0}^{k=\infty} (-1)^k \frac{1.3\ldots(2k-1)}{2.4\ldots 2k} P^{-\frac{2k+1}{2}} Q^k.$$

10. In order that this development of $\frac{1}{\Delta}$ in a series of ascending powers of Q, or, if one likes, of $\sin^2 \frac{1}{2} I$, may be legitimate, it is necessary that the elements of the orbits should be such that the numerical value of $\frac{Q}{P}$ should be always less than unity. P is the square of the distance of the two planets after the plane of the orbit of one has been brought into coincidence with the plane of the other by revolving it about the line of intersection of the two planes. Taking then a system of rectangular axes passing through the centre of the sun, and directing the axis of x along the line of intersection, it is plain the equations of the orbits may be written

$$\sqrt{(x^2 + y^2)} + \alpha x + \beta y = p,$$
$$\sqrt{(x'^2 + y'^2)} + \alpha' x' + \beta' y' = p',$$

α, β, α', β' being constants. And the variables x, y, x', y' satisfying these equations, the question depends on the finding of the values of them which render the expression

$$D = \frac{yy'}{(x-x')^2 + (y-y')^2}$$

a maximum or a minimum. According to the known theory of maxima and minima, the equations, which, in combination with the equations of the orbits, give these values, are

$$-2D(x-x') + \mu\left[\frac{x}{\sqrt{(x^2+y^2)}} + \alpha\right] = 0,$$

$$2D(x-x') + \mu'\left[\frac{x'}{\sqrt{(x'^2+y'^2)}} + \alpha'\right] = 0,$$

$$y' - 2D(y-y') + \mu\left[\frac{y}{\sqrt{(x^2+y^2)}} + \beta\right] = 0,$$

$$y + 2D(y-y') + \mu'\left[\frac{y'}{\sqrt{(x'^2+y'^2)}} + \beta'\right] = 0,$$

where μ and μ' are the multipliers of the partial derivatives of the two equations of condition. A complete investigation of this question would be conducted in the following manner. Eliminate from the seven equations last given the six quantities x, y, x', y', μ, μ'; the result will be an algebraical equation determining the unknown D Having derived the Sturmian functions of this, one will ascertain by the substitution of the values $D = \frac{1}{4\sin^2\frac{1}{2}I}$, $D = +\infty$, and again of $D = -\frac{1}{4\sin^2\frac{1}{2}I}$, $D = -\infty$, whether any roots lie between these limits; if none, $\frac{1}{\Delta}$ can be expanded in a series of ascending powers of $\sin^2\frac{1}{2}I$, in the contrary case not. In this way we shall arrive at the condition or conditions necessary and sufficient for the legitimacy of this expansion.

11. This procedure would doubtless lead to very complicated formulæ, hence we are obliged to pass over it. However equations can be readily got, which, by a tentative process, afford the maxima and minima values of D. Multiply the four equations last given respectively by x, x', y, y' and add the resulting equations, having regard to the equations of the orbits and the value of D, we thus arrive at the simple relation

$$p\mu + p'\mu' = 0.$$

Putting, for simplicity, $x = r\cos\theta$, $x' = r'\cos\theta'$, the addition of the first and second of the same group of four equations gives

$$\mu(\cos\theta + \alpha) + \mu'(\cos\theta' + \alpha') = 0.$$

By combining this with the preceding is obtained

$$\frac{\cos\theta' + \alpha'}{p'} = \frac{\cos\theta + \alpha}{p}.$$

Again the addition of the same equations, multiplied severally by x, $-x'$, y, $-y'$, gives the equation

$$2D\,(r'^2 - r^2) = p'\mu' - p\mu.$$

Dividing the left member of this by $2D\,(x' - x)$, and the terms of the right member by its equivalents derived from the first and second equations, we get

$$\frac{r'^2 - r^2}{x' - x} = \frac{p'}{\cos\theta' + \alpha'} + \frac{p}{\cos\theta + \alpha'},$$

or

$$\frac{r'\cos\theta' - r\cos\theta}{r'^2 - r^2} = \frac{\cos\theta + \alpha}{2p}.$$

This and the equation

$$\frac{\cos\theta' + \alpha'}{p'} = \frac{\cos\theta + \alpha}{p}$$

determine the values of the variables θ and θ' which render D a maximum or minimum. When the orbits are nearly circular these values are in the neighborhood of $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$. When both orbits are circles the solution is very simple, and we find that in order the development may be legitimate, we must have

$$\sin\frac{I}{2} < \frac{a' - a}{2\sqrt{(aa')}},$$

a and a' being the mean distances of the planets from the sun.

12. Assuming that this development is legitimate, we have to develop $P^{-\frac{2k+1}{2}}Q^k$ in terms of s and s'. We have

$$r\cos v = a\,(\cos u - e) = \frac{a}{2}\left(s + \frac{1}{s} - 2e\right),$$

$$r\sin v = a\sqrt{(1-e^2)}\sin u = \frac{a}{2\sqrt{-1}}\sqrt{(1-e^2)}\left(s - \frac{1}{s}\right),$$

whence

$$r\cos v + r\sin v\,\sqrt{-1} = r\varepsilon^{v\sqrt{-1}}$$

$$= a\left[\frac{1+\sqrt{(1-e^2)}}{2}s + \frac{1-\sqrt{(1-e^2)}}{2}\frac{1}{s} - e\right],$$

and by putting

$$\frac{1+\sqrt{(1-e^2)}}{2} = \eta, \quad \frac{e}{1+\sqrt{(1-e^2)}} = \omega,$$

we get

$$r\varepsilon^{v\sqrt{-1}} = a\eta s\left(1 - \frac{\omega}{s}\right)^2.$$

And the value of $r\varepsilon^{-v\sqrt{-1}}$ is evidently obtained by substituting in this $\frac{1}{s}$ for s, hence
$$r\varepsilon^{-v\sqrt{-1}} = a\eta\frac{1}{s}(1-\omega s)^2.$$

From these two equations may be derived
$$r = a\eta(1-\omega s)\left(1-\frac{\omega}{s}\right),$$
$$\varepsilon^{v\sqrt{-1}} = \frac{s-\omega}{1-\omega s}.$$

Writing γ for $\Pi - \Pi'$, we have
$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \left[1 - 2\frac{r}{r'}\cos(v-v'+\gamma) + \frac{r^2}{r'^2}\right]^{-\frac{2k+1}{2}}.$$

The right member of this is developable in a series of integral powers of the exponential $\varepsilon^{(v-v'+\gamma)\sqrt{-1}}$ when $\frac{r}{r'}$ is always less than unity. This condition is fulfilled when we have $a(1+e) < a'(1-e')$. Writing g for $\varepsilon^{\gamma\sqrt{-1}}$, let
$$(r'^{-2}P)^{-\frac{2k+1}{2}} = \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} . B^{(j)}_{\frac{2k+1}{2}}\,\varepsilon^{j(v-v'+\gamma)\sqrt{-1}}$$
$$= \tfrac{1}{2}\sum_{j=-\infty}^{j=+\infty} . B^{(j)}_{\frac{2k+1}{2}}\left(\frac{s-\omega}{1-\omega s}\right)^{j}\left(\frac{s'-\omega'}{1-\omega' s'}\right)^{-j} g^j.$$

$B^{(j)}_{\frac{2k+1}{2}}$ is the same function of $\frac{r}{r'}$ that Laplace's $b^{(j)}_{\frac{2k+1}{2}}$ is of $\frac{a}{a'} = \alpha$. The approximate value of $\frac{r}{r'}$ being α, any function of $\frac{r}{r'}$ can be expanded in a series of ascending powers of $\frac{r}{r'} - \alpha$ by Taylor's Theorem. And as we have
$$\frac{r}{r'} - \alpha = \alpha\left\{\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - 1\right\},$$
consequently
$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty}\frac{1}{n!}\alpha^n\frac{d^n b^{(j)}_{\frac{2k+1}{2}}}{d\alpha^n}\left\{\frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} - 1\right\},$$
n being an integer, and $n!$ denoting the product of all integers up to n inclusive, it being understood that $0! = 1$. Expanding the last factor of this expression by the binomial theorem, and employing the notation $[i, j]$ for the coefficient of x^j in the expansion of $(1+x)^i$, we have, p being an integer,

$$B^{(j)}_{\frac{2k+1}{2}} = \sum_{n=0}^{n=\infty} \sum_{p=0}^{p=n} (-1)^{n-p} \frac{[n, p]}{n!} a_1^{n} \frac{d^n b^{(j)}_{\frac{2n+1}{2}}}{da^n} \left\{ \frac{\eta(1-\omega s)\left(1-\frac{\omega}{s}\right)}{\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right)} \right\}^{p}.$$

13. In the next place the development of Q in terms of s and s' must be formed. We have

$$r \sin(v + \Pi) = \frac{1}{2\sqrt{-1}} \left[r\varepsilon^{(v+\Pi)\sqrt{-1}} - r\varepsilon^{-(v+\Pi)\sqrt{-1}} \right],$$

$$r' \sin(v' + \Pi') = \frac{1}{2\sqrt{-1}} \left[r'\varepsilon^{(v'+\Pi')\sqrt{-1}} - r'\varepsilon^{-(v'+\Pi')\sqrt{-1}} \right],$$

and putting $\qquad \Pi + \Pi' = \theta, \qquad h = \varepsilon^{\theta\sqrt{-1}},$

we find

$$Q = -aa'\eta\eta' \sin^2\frac{I}{2} \cdot \left[s\left(1-\frac{\omega}{s}\right)^2 - \frac{1}{s}\left(1-\omega s\right)^2 g^{-1}h^{-1} \right]$$
$$\times \left[s'\left(1-\frac{\omega'}{s'}\right)^2 h - \frac{1}{s'}\left(1-\omega' s'\right)^2 g \right].$$

Raising this expression to the k^{th} power, and multiplying by

$$r'^{-(2k+1)} = \left[a'\eta'(1-\omega' s')\left(1-\frac{\omega'}{s'}\right) \right]^{-(2k+1)},$$

we find that the part of $r'^{-(2k+1)}\, Q^k$ which has $h^{i'''}$ as a factor is

$$\frac{1}{a'} a^k \eta^k \eta'^{-(k+1)} \sin^{2k}\frac{I}{2} \sum_{n'=0}^{n'=k-i'''} (-1)^{i'''} [k, n'] [k, k-i'''-n']$$
$$\times s^{2i'''-k+2n'} (1-\omega s)^{2k-2i'''-2n'} \left(1-\frac{\omega}{s}\right)^{2i'''+2n'}$$
$$\times s'^{k-2n'} (1-\omega' s')^{2n'-2k-1} \left(1-\frac{\omega'}{s'}\right)^{-2n'-1} g^{i'''-k+2n'} h^{i'''}.$$

14. We are now in the possession of all the developments necessary for exhibiting the function $\frac{1}{\Delta}$ in terms of s and s'. In order to obtain the part of this function which has $g^{i''}h^{i'''}$ for a factor, we must put, in the formulæ of § 12,

$$j = i'' - i''' + k - 2n',$$

and the chief operation here is the addition of the exponents of the quantities s, $1-\omega s$, $1-\frac{\omega}{s}$, and the similar functions of s' which are found in the three formulæ for $(r'^{-2}P)^{-\frac{2k+1}{2}}$, $B^{(j)}_{\frac{2k+1}{2}}$ and $r'^{-(2k+1)}\, Q^k$. For brevity we will write

$$[k] = \frac{1.3\ldots(2k-1)}{2.4\ldots.2k}.$$

Then the part of $\frac{1}{\Delta}$,which has $g^{i''}h^{i'''}$ for a factor, is

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\ \sum_{n'=0}^{n'=k-i'''}\ \sum_{n=0}^{n=\infty}\ \sum_{p=0}^{p=n}\ (-1)^{k-i'''+n-p}\ \frac{[k]\,[k,\,n']\,[k,\,k-i'''-n']\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n b^{(i''-i'''+k-2n')}_{\frac{2k+1}{2}}}{da^n}\qquad \sin^{2k}\frac{I}{2}$$

$$\times \eta^{k+p}\, s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$

$$\times \eta'^{-k-p-1}\, s'^{-i''+i'''}\,(1-\omega' s')^{-k-p-1+i''-i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''}\qquad g^{i''}h^{i'''}.$$

We observe that in this expression the summation with respect to n' affects only the integral coefficients $[k,\,n']$, $[k,\,k-i'''-n']$ and the upper index of the quantity b, hence if a new function of a is assumed, which is a linear function of the b's, and such that

$$\boldsymbol{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}=\sum_{n'=0}^{n'=k-i'''}[k,\,n']\,[k,\,k-i'''-n']\ b^{(i''-i'''+k-2n')}_{\frac{2k+1}{2}}\ .$$

it will take the following simpler form

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\ \sum_{n=0}^{n=\infty}\ \sum_{p=0}^{p=n}\ (-1)^{k-i'''+n-p}\ \frac{[k]\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n \boldsymbol{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\quad \sin^{2k}\frac{I}{2}$$

$$\times \eta^{k+p}\, s^{i''+i'''}(1-\omega s)^{k+p-i''-i'''}\left(1-\frac{\omega}{s}\right)^{k+p+i''+i'''}$$

$$\times \eta'^{-k-p-1}\, s'^{-i''+i'''}(1-\omega' s')^{-k-p-1+i''-i'''}\left(1-\frac{\omega'}{s'}\right)^{-k-p-1-i''+i'''}\qquad g^{i''}h^{i'''}.$$

15. In order to get the coefficient of $z^i\, z'^{\,i'}$ in the expansion of $\frac{1}{\Delta}$, according to the foregoing investigation we must multiply the preceding expression by

$$\frac{r\,r'}{a\,a'}\ \varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)+\frac{i'e'}{2}\left(s'-\frac{1}{s'}\right)}.$$

Hence, if for brevity, we adopt the functional notation

$$S\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix}=\eta^i\, s^j\,(1-\omega\, s)^{i-j}\left(1-\frac{\omega}{s}\right)^{i+j}\ \varepsilon^{\frac{ke}{2}\left(s-\frac{1}{s}\right)},$$

$$S'\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix}=\eta'^{\,i} s'^{\,j}\,(1-\omega' s')^{i-j}\left(1-\frac{\omega'}{s'}\right)^{i+j}\ \varepsilon^{\frac{ke'}{2}\left(s'-\frac{1}{s'}\right)},$$

the coefficient of $z^i\, z^{i'}\, g^{i''}\, h^{i'''}$ in $\frac{1}{\Delta}$ will be equal to the coefficient of $s^i\, s'^{\,i'}$ in

$$\frac{1}{2a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k]\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n\mathcal{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\,.\;S\begin{pmatrix}k+p\\ i''+i'''\\ i\end{pmatrix}.\,S'\begin{pmatrix}-(k+p+1)\\ -(i''-i''')\\ i'\end{pmatrix}.$$

If then the coefficient of s^i in the expansion of S is denoted by $\mathcal{E}$ followed by the same indices, and the coefficient of $s'^{i'}$ in the expansion of S' by $\mathcal{E}'$ in like manner, $\mathcal{E}$ will be a function of e only, and $\mathcal{E}'$ a function of e' only; and, it being understood that each argument is taken but once, that is the negative of the argument is not considered, the coefficient of

$$\cos\,(\,i\zeta\;+\;i'\zeta'\;+\;i''\gamma\;+\;i'''\theta\,)$$

in the expansion of $\frac{1}{\Delta}$ is expressed thus

$$\frac{1}{a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k]\,[n,\,p]}{n\,!}$$

$$\times a^{k+n}\frac{d^n\mathcal{B}^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\cdot\mathcal{E}\begin{pmatrix}k+p\\ i''+i'''\\ i\end{pmatrix}\cdot\mathcal{E}'\begin{pmatrix}-(k+p+1)\\ -(i''-i''')\\ i'\end{pmatrix}.$$

As in this formula k ought to be a positive integer, it will prevent embarrassment, if the arguments are so taken that i''' may not be negative. In the case where i, i', i'' and i''' are all zero, the expression must be divided by 2.

16. Thus we have arrived at an expression for the general coefficient involving only three signs of summation; and it may be remarked that all the coefficients are exhibited in precisely similar forms. Thus, to pass from one argument to an other, we have only to make the suitable changes in the two lower indices of the functions $\mathcal{E}$ and $\mathcal{E}'$ and in the upper indices of $\mathcal{B}$, and commence the summation with reference to k with the new value of i''' instead of the old. Hence, from this expression, we can write out a scheme or blank form, which, when the indices proper to the argument are filled in, will be the coefficient of the cosine of it in the expansion of $\frac{1}{\Delta}$. Such a blank form is written below; the indices i'' and i''' are omitted from $\mathcal{B}$, and the two lower indices from $\mathcal{E}$ and $\mathcal{E}'$, and the upper indices of these quantities, for the sake of facility in writing, are placed to the right and at the foot. The factor $\frac{1}{a'}$, common to the whole expresson, is also omitted, so that the formula gives the coefficient in the expansion of $\frac{a'}{\Delta}$. In making use of it, one must commence at the portion which has $\sin^{2i'''}\frac{1}{2}I$ for a factor, all the preceding parts being supposed to be suppressed. As to the double sign

before these factors, the upper or lower is taken according as i''' is even or odd. It is hoped that a sufficient number of terms have been written to render the law evident, so that they may be continued as far as desired.

$$\begin{aligned}
& b_{\frac{1}{2}} E_{0} E'_{-1} \\
& -\frac{1}{1}\,\alpha \frac{db_{\frac{1}{2}}}{d\alpha}\Big[E_{0}E'_{-1} - E_{1}E'_{-2}\Big] \\
& +\frac{1}{1.2}\,\alpha^{2}\frac{d^{2}b_{\frac{1}{2}}}{d\alpha^{2}}\Big[E_{0}E'_{-1} - 2E_{1}E'_{-2} + E_{2}E'_{-3}\Big] \\
& -\frac{1}{1.2.3}\,\alpha^{3}\frac{d^{3}b_{\frac{1}{2}}}{d\alpha^{3}}\Big[E_{0}E'_{-1} - 3E_{1}E'_{-2} + 3E_{2}E'_{-3} - E_{3}E'_{-4}\Big] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$\pm\frac{1}{2}\sin^{2}\frac{I}{2}\left\{\begin{aligned}
& \alpha B_{\frac{3}{2}} E_{1}E'_{-2} \\
& -\frac{1}{1}\,\alpha^{2}\frac{dB_{\frac{3}{2}}}{d\alpha}\Big[E_{1}E'_{-2} - E_{2}E'_{-3}\Big] \\
& +\frac{1}{1.2}\,\alpha^{3}\frac{d^{2}B_{\frac{3}{2}}}{d\alpha^{2}}\Big[E_{1}E'_{-2} - 2E_{2}E'_{-3} + E_{3}E'_{-4}\Big] \\
& -\frac{1}{1.2.3}\,\alpha^{4}\frac{d^{3}B_{\frac{3}{2}}}{d\alpha^{3}}\Big[E_{1}E'_{-2} - 3E_{2}E'_{-3} + 3E_{3}E'_{-4} - E_{4}E'_{-5}\Big] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}$$

$$\pm\frac{1.3}{2.4}\sin^{4}\frac{I}{2}\left\{\begin{aligned}
& \alpha^{2} B_{\frac{5}{2}} E_{2}E'_{-3} \\
& -\frac{1}{1}\,\alpha^{3}\frac{dB_{\frac{5}{2}}}{d\alpha}\Big[E_{2}E'_{-3} - E_{3}E'_{-4}\Big] \\
& +\frac{1}{1.2}\,\alpha^{4}\frac{d^{2}B_{\frac{5}{2}}}{d\alpha^{2}}\Big[E_{2}E'_{-3} - 2E_{3}E'_{-4} + E_{4}E'_{-5}\Big] \\
& -\frac{1}{1.2.3}\,\alpha^{5}\frac{d^{3}B_{\frac{5}{2}}}{d\alpha^{3}}\Big[E_{2}E'_{-3} - 3E_{3}E'_{-4} + 3E_{4}E'_{-5} - E_{5}E'_{-6}\Big] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}$$

$$\pm\frac{1.3.5}{2.4.6}\sin^{6}\frac{I}{2}\left\{\begin{aligned}
& \alpha^{3} B_{\frac{7}{2}} E_{3}E'_{-4} \\
& -\frac{1}{1}\,\alpha^{4}\frac{dB_{\frac{7}{2}}}{d\alpha}\Big[E_{3}E'_{-4} - E_{4}E'_{-5}\Big] \\
& +\frac{1}{1.2}\,\alpha^{5}\frac{d^{2}B_{\frac{7}{2}}}{d\alpha^{2}}\Big[E_{3}E'_{-4} - 2E_{4}E'_{-5} + E_{5}E'_{-6}\Big] \\
& -\frac{1}{1.2.3}\,\alpha^{6}\frac{d^{3}B_{\frac{7}{2}}}{d\alpha^{3}}\Big[E_{3}E'_{-4} - 3E_{4}E'_{-5} + 3E_{5}E'_{-6} - E_{6}E'_{-7}\Big] \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}$$

$$\pm \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

For illustration, let it be desired to obtain the coefficient of

$$\cos(2\zeta - 5\zeta' + 2\gamma),$$

from which arises the larger part of the great inequality of Jupiter and Saturn, we have only to imagine that the lower indices $\begin{pmatrix} \cdot \\ 2 \\ 2 \end{pmatrix}$ are every where applied to $\mathfrak{E}$, and the indices $\begin{pmatrix} \cdot \\ -2 \\ -5 \end{pmatrix}$ to $\mathfrak{E}'$, the indices $(2, 0)$ to $\mathfrak{B}$; and as we have $i''' = 0$, we suppress nothing and take the upper of the double signs.

17. The quantities $\mathfrak{B}$ are very simply expressed in terms of the b's, the following are all that are needed when terms of the eighth order with respect to the inclination of the orbits are neglected.

$$\begin{aligned}
\mathfrak{B}_{\frac{1}{2}}^{(i,0)} &= b_{\frac{1}{2}}^{(i)}, \\
\mathfrak{B}_{\frac{3}{2}}^{(i,0)} &= b_{\frac{3}{2}}^{(i+1)} + b_{\frac{3}{2}}^{(i-1)}, \\
\mathfrak{B}_{\frac{3}{2}}^{(i,1)} &= b_{\frac{3}{2}}^{(i)}, \\
\mathfrak{B}_{\frac{5}{2}}^{(i,0)} &= b_{\frac{5}{2}}^{(i+2)} + 4b_{\frac{5}{2}}^{(i)} + b_{\frac{5}{2}}^{(i-2)}, \\
\mathfrak{B}_{\frac{5}{2}}^{(i,1)} &= 2b_{\frac{5}{2}}^{(i+1)} + 2b_{\frac{5}{2}}^{(i-1)}, \\
\mathfrak{B}_{\frac{5}{2}}^{(i,2)} &= b_{\frac{5}{2}}^{(i)}, \\
\mathfrak{B}_{\frac{7}{2}}^{(i,0)} &= b_{\frac{7}{2}}^{(i+3)} + 9b_{\frac{7}{2}}^{(i+1)} + 9b_{\frac{7}{2}}^{(i-1)} + b_{\frac{7}{2}}^{(i-3)}, \\
\mathfrak{B}_{\frac{7}{2}}^{(i,1)} &= 3b_{\frac{7}{2}}^{(i+2)} + 9b_{\frac{7}{2}}^{(i)} + 3b_{\frac{7}{2}}^{(i-2)}, \\
\mathfrak{B}_{\frac{7}{2}}^{(i,2)} &= 3b_{\frac{7}{2}}^{(i+1)} + 3b_{\frac{7}{2}}^{(i-1)}, \\
\mathfrak{B}_{\frac{7}{2}}^{(i,3)} &= b_{\frac{7}{2}}^{(i)}
\end{aligned}$$

18. In computing the factors of the preceding formula which depend on $\mathfrak{E}$ and $\mathfrak{E}'$, the following abbreviation can be used. M_n denoting the factor which multiplies $= \frac{1}{n!}\, a^{k+n} \frac{d^n \mathfrak{B}_{\frac{2k+1}{2}}}{da^n}$, and Δ being the symbol of finite differences with respect to n, it is plain that

$$\Delta^n M_n = (-1)^n \mathfrak{E}_{k+n}\, \mathfrak{E}'_{-(k+n+1)}.$$

Hence if the products $\mathfrak{E}_{k+n}\, \mathfrak{E}'_{-(k+n+1)}$ are computed for the various values of n, and are taken alternately with the positive and negative sign, and are written as if they were the successive differences of a function, we shall get the values of the factors M_n by filling out the scheme of differences. This abbreviation is applicable equally whether we are making a numerical computation of the coefficient or a literal one. In the latter case the abbreviation can be applied separately to each term of the form $Ce^{i}e'^{i'}$ in the products $\mathfrak{E}_k\mathfrak{E}'_{-(k+1)}$.

19. We proceed now to discuss the functions $\mathfrak{E}$. From their definition we have

$$\left(\frac{r}{a}\right)^{i} \varepsilon^{jv\sqrt{-1}} = \sum_{k=-\infty}^{k=+\infty} \mathfrak{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} z^k,$$

whence

$$\left(\frac{r}{a}\right)^{i} \cos jv = \tfrac{1}{2} \sum_{k=\ \infty}^{k=+\infty}{}' \left[\mathfrak{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} + \mathfrak{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix}\right] \cos k\zeta,$$

$$\left(\frac{r}{a}\right)^{i} \sin jv = \tfrac{1}{2} \sum_{k=-\infty}^{k=+\infty} \left[\mathfrak{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} - \mathfrak{E}\begin{pmatrix} i \\ -j \\ k \end{pmatrix}\right] \sin k\zeta.$$

From which we gather that the functions $\mathfrak{E}$ can be computed by definite integrals thus

$$\mathfrak{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \frac{1}{\pi}\int_0^{\pi} \left(\frac{r}{a}\right)^{i} \cos(jv - k\zeta)d\zeta.$$

Let us now suppose that the coefficient of s^k, in the expansion of

$$\eta^i s^j (1 - \omega s)^{i-j}\left(1 - \frac{\omega}{s}\right)^{i+j}$$

in powers of s, is denoted by $E\begin{pmatrix} i \\ j \\ k \end{pmatrix}$, then evidently

$$\mathfrak{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix} = \sum_{l=-\infty}^{l=+\infty} . E\begin{pmatrix} i+1 \\ j \\ k-l \end{pmatrix} J^{(l)}_{\frac{ke}{2}}.$$

By writing in the expression $1 \div s$ for s and changing the sign of j, it remains unaltered; hence the relation

$$E\begin{pmatrix} i \\ -j \\ -k \end{pmatrix} = E\begin{pmatrix} i \\ j \\ k \end{pmatrix}.$$

By developing the factors of the expression

$$\eta^i s^j (1 - \omega s)^{i-j}\left(1 - \frac{\omega}{s}\right)^{i+j}.$$

by the binomial theorem, we easily get

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j}[i-j,\ k-j]\ \eta^i\ \omega^{k-j}$$

$$t\times\left[1 + \frac{(i+j)\,(i-k)}{1.(k-j+1)}\,\omega^2 + \frac{(i+j)(i+j-1)(i-k)(i-k-1)}{1.2.\,(k-j+1)\,(k-j+2)}\omega^4 + \ldots\right].$$

This equation, as written, is correct only when $k-j$ is not negative, but by h e relation given above we can reduce the case of $k-j$ negative to that where it is positive. The factor in the brackets is a case of the series

$$1 + \frac{\alpha\,.\,\beta}{1\,.\,\gamma}\,x + \frac{x(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,.\,\gamma\,.\,(\gamma+1)}\,x^2 + \ldots$$

treated by Gauss in a memoir entitled "*Disquisitiones generales circa seriem infinitam &c.*" (See Gauss' Werke, Vol. III. p. 123, and especially the "Nachlass." p.207). According to Gauss' notation

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k\ j}[i-j,\ k-j]\,\eta^i\omega^{k-j}F(-i-j,\ k-i,\ k-j+1,\ \omega^2).$$

Whenever, of $i+j$ and $i-j$, one is not negative, this series terminates after a certain number of terms, thus affording a finite expression for the function. But when these integers are both negative, the series is infinite. However it can be easily transformed into another which like the former is finite. From Gauss' investigation of these series, (See the volume just quoted, p. 209, Equation [82]), we have

$$F(\alpha,\ \beta,\ \gamma,\ x) = (1-x)^{\gamma-\alpha-\beta}\ F(\gamma-\alpha,\ \gamma-\beta,\ \gamma,\ x).$$

Applying this to our expression, we get

$$E\begin{pmatrix} i \\ j \\ k \end{pmatrix} = (-1)^{k-j}[i-j,\ k-j]\eta^i\omega^{k-j}(1-\omega^2)^{2i+1}$$

$$\times F(i+k+1,\ i-j+1,\ k-j+1,\ \omega^2).$$

This expression is evidently finite when $i-j$ and $i+j$ are negative.

20. The developments of the functions $\mathcal{E}$ in powers of e as far as e^7 have been tabulated by Prof. Cayley in the Memoirs of the Royal Astronomical Society, Vol. XXVII. It would conduce to the ready employment of the preceding formulæ if we had the function $\mathcal{E}\begin{pmatrix} i \\ j \\ k \end{pmatrix}$ explicitly expanded in ascending powers of e, but the attempts I have made to write such a series lead to extremely complex forms of the coefficients. Hence I shall give here only the coefficient of the lowest power of e in this function, which suffices for obtaining all the terms of the lowest order in any coefficient of the expansion of $1\div\Delta$. We have when $j-k$ is positive,

$$\mathcal{E}\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \Big[[i+j, j-k]+[i+j, j-k-1]\frac{k}{1}+[i+j, j-k-2]\frac{k^2}{1.2}$$

$$+\ \ldots\ldots + [i+j,\ 0]\frac{k^{j-k}}{(j-k)!}\Big]\left(-\frac{e}{2}\right)^{j-k},$$

and when $k-j$ is positive,

$$\mathcal{E}\begin{pmatrix} i-1 \\ j \\ k \end{pmatrix} = \Big[[i-j,\ k-j]-[i-j,\ k-j-1]\frac{k}{1}+[i-j,\ k-j-2]\frac{k^2}{1.2}$$

$$-\ \ldots\ldots + [i-j,\ 0]\frac{(-k)^{k-j}}{(k-j)!}\Big]\left(-\frac{e}{2}\right)^{k-j}.$$

21. Thus in the example alluded to above of the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$, we find that the terms of the lowest order in $\mathcal{E}$ and $\mathcal{E}'$, (omitting here, as in the scheme, the two lower indices), are

$$\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = 1,$$

$$\mathcal{E}'_{-1} = -\left[[-2,3] - [-2,2]\frac{5}{1} + [-2,1]\frac{5^2}{1.2} - [-2,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{389}{48}e'^3,$$

$$\mathcal{E}'_{-2} = -\left[[-3,3] - [-3,2]\frac{5}{1} + [-3,1]\frac{5^2}{1.2} - [-3,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{590}{48}e'^3,$$

$$\mathcal{E}'_{-3} = -\left[[-4,3] - [-4,2]\frac{5}{1} + [-4,1]\frac{5^2}{1.2} - [-4,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{845}{48}e'^3,$$

$$\mathcal{E}'_{-4} = -\left[[-5,3] - [-5,2]\frac{5}{1} + [-5,1]\frac{5^2}{1.2} - [-5,0]\frac{5^3}{1.2.3}\right]\left(\frac{e'}{2}\right)^3 = \frac{1160}{48}e'^3.$$

Bringing into use our method of abbreviation, we multiply each of the preceding numerical coefficients by 48 in order to avoid fractions, and then write them alternately with the positive and negative signs in a diagonal line, and from these, as successive orders of differences, derive the numbers standing in the vertical columns, thus,

$$\begin{array}{rrrrr}
+389 & & & & \\
 & -590 & & & \\
-201 & & +845 & & \\
 & +255 & & -1160 & \\
+54 & & -315 & & . \\
 & -60 & & +381 & \\
-6 & & +66 & & . \\
 & +6 & & -72 & \\
0 & & -6 & & . \\
 & 0 & & +6 & \\
 & & 0 & & . \\
 & & & 0 &
\end{array}$$

and dividing the numbers of the first column respectively by 1, —1, 1 . 2, — 1.2.3, we get the following as the terms of the lowest order in the coefficient of $\cos(2\zeta - 5\zeta' + 2\gamma)$ in $a' \div \Delta$,

$$\frac{1}{48}\left[389\, b^{(2)}_{\frac{1}{2}} + 201\, a\frac{db^{(2)}_{\frac{1}{2}}}{da} + 27a^2\frac{d^2b^{(2)}_{\frac{1}{2}}}{da^2} + a^3\frac{d^3b^{(2)}_{\frac{1}{2}}}{da^3}\right]e'^3,$$

which agrees with that found in the books. The following additional terms of the same coefficient can be written from the second, third, &c., columns, viz, those which are multiplied by e'^3 and the various powers of $\sin^2\frac{1}{2}I$,

$$-\frac{1}{2}\,\frac{1}{48}\left[590a\,\mathcal{B}^{(2,0)}_{\frac{3}{2}} + 255a^2\frac{d\mathcal{B}^{(2,0)}_{\frac{3}{2}}}{da} + 30a^3\frac{d^2\mathcal{B}^{(2,0)}_{\frac{3}{2}}}{da^2} + a^4\frac{d^3\mathcal{B}^{(2,0)}_{\frac{3}{2}}}{da^3}\right]e'^3\sin^2\frac{I}{2}$$

$$+\frac{1.3}{2.4}\,\frac{1}{48}\left[845a^2\mathcal{B}^{(2,0)}_{\frac{5}{2}} + 315a^3\frac{d\mathcal{B}^{(2,0)}_{\frac{5}{2}}}{da} + 33a^4\frac{d^2\mathcal{B}^{(2,0)}_{\frac{5}{2}}}{da^2} + a^5\frac{d^3\mathcal{B}^{(2,0)}_{\frac{5}{2}}}{da^3}\right]e'^3\sin^4\frac{I}{2}$$

— &c. . . . .

22. When we wish to obtain only the terms independent of ζ and ζ', that is those on which the secular perturbations depend, $i=0$ and $i'=0$, and the Besselian function J disappears from the expressions giving the values of E and E', and the coefficient of $\cos(i''\gamma + i'''\theta)$ in the expansion of $\frac{1}{\Delta}$ can be written

$$\frac{1}{a'}\sum_{k=i'''}^{k=\infty}\sum_{n=0}^{n=\infty}\sum_{p=0}^{p=n}(-1)^{k-i'''+n-p}\frac{[k][n,p]}{n!}$$
$$\times a^{k+n}\frac{d^n B^{(i'',\,i''')}_{\frac{2k+1}{2}}}{da^n}\sin^{2k}\frac{I}{2}\,.\,E\begin{pmatrix}k+p+1\\ i''+i'''\\ 0\end{pmatrix}.\,E'\begin{pmatrix}-(k+p)\\ -(i''-i''')\\ 0\end{pmatrix}.$$

23. In leaving the subject of the development of $1\div\Delta$, it may be well to note that two other forms can be given to the expression of the general coefficient, by employing, instead of the expression given above, either of the following

$$\frac{r}{r'} - a = a\,\frac{\frac{e'}{2}\left(s'+\frac{1}{s'}\right)-\frac{e}{2}\left(s+\frac{1}{s}\right)}{1-\frac{e'}{2}\left(s'+\frac{1}{s'}\right)}$$
$$= a\,\frac{\frac{e'}{2}\left(s'+\frac{1}{s'}\right)-\frac{c}{2}\left(s+\frac{1}{s}\right)}{\eta'(1-\omega's')\left(1-\frac{\omega'}{s'}\right)}.$$

But as they do not possess as much symmetry and brevity as the form given above, we will pass over them.

24. The second part of the Perturbative Function ,omitting the factor m', is

$$-\frac{r}{r'^2}\cos\psi = -\frac{r}{r'^2}\left[\cos^2\frac{I}{2}\cos(v-v'+\gamma)+\sin^2\frac{I}{2}\cos(v+v'+\theta)\right]$$
$$= -\frac{1}{2}\,\frac{r}{r'^2}\cos^2\frac{I}{2}\left[g\varepsilon^{(v-v')\sqrt{-1}} + g^{-1}\varepsilon^{-(v-v')\sqrt{-1}}\right]$$
$$-\frac{1}{2}\,\frac{r}{r'^2}\sin^2\frac{I}{2}\left[h\varepsilon^{(v+v')\sqrt{-1}} + h^{-1}\varepsilon^{-(v+v')\sqrt{-1}}\right].$$

According to the first theorem of § 3, the coefficient of z^0 in $\frac{r}{a}\varepsilon^{v\sqrt{-1}}$ is equal to that of s^0 in

$$\eta^2 s(1-\omega s)\left(1-\frac{\omega}{s}\right)^3;$$

or it is equal to $\qquad -3\eta^2\omega(1+\omega^2) = -\frac{3}{2}e.$

And according to the second theorem, the coefficient of z^i in the same function is equal to that of s^i in

$$\frac{\eta}{i}\,\frac{d}{ds}\left[s\left(1-\frac{\omega}{s}\right)^2\right].\,s\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$
$$=\frac{\eta}{i}\left(s-\frac{\omega^2}{s}\right)\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}$$

Hence we have

$$\frac{r}{a}\,\varepsilon^{v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}{}'\,\frac{\eta}{i}\left[J^{(i-1)}_{\frac{ie}{2}}-\omega^2 J^{(i+1)}_{\frac{ie}{2}}\right]z^i.$$

And by simply writing $1\div z$ for z,

$$\frac{r}{a}\,\varepsilon^{-v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}{}'\,\frac{\eta}{i}\left[\omega^2 J^{(i-1)}_{\frac{ie}{2}}-J^{(i+1)}_{\frac{ie}{2}}\right]z^i.$$

The well known differential equations of elliptic motion

$$\frac{d^2x}{d\zeta^2}+\frac{a^3}{r^3}x=0,$$
$$\frac{d^2y}{d\zeta^2}+\frac{a^3}{r^3}y=0,$$

supposing the axis of x to be directed towards the perihelion, give us the equation

$$\frac{a^2}{r^2}\,\varepsilon^{v\sqrt{-1}}=-\frac{d^2\left(\frac{r}{a}\varepsilon^{v\sqrt{-1}}\right)}{d\zeta^2},$$

and consequently these two

$$\frac{a^2}{r^2}\varepsilon^{v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}{}'\,i\eta\left[J^{(i-1)}_{\frac{ie}{2}}-\omega^2 J^{(i+1)}_{\frac{ie}{2}}\right]z^i,$$
$$\frac{a^2}{r^2}\varepsilon^{-v\sqrt{-1}}=\sum_{i=-\infty}^{i=+\infty}{}'\,i\eta\left[\omega^2 J^{(i-1)}_{\frac{ie}{2}}-J^{(i+1)}_{\frac{ie}{2}}\right]z^i.$$

By substituting these values in the expression given above for $-\frac{r}{r'^2}\cos\psi$, it is not difficult to see that, in it, the coefficient of

$$\cos(i\zeta+i'\zeta'+\gamma)$$

is $$-\frac{a}{a'^2}\cos^2\frac{I}{2}\,.\,\frac{\eta}{i}\left[J^{(i-1)}_{\frac{ie}{2}}-\omega^2 J^{(i+1)}_{\frac{ie}{2}}\right].\,i'\eta'\left[\omega'^2 J^{(i'-1)}_{\frac{i'e'}{2}}-J^{(i'+1)}_{\frac{i'e'}{2}}\right],$$

and the coefficient of

$$\cos(i\zeta+i'\zeta'+\theta)$$

is $$-\frac{a}{a'^2}\sin^2\frac{I}{2}\,.\,\frac{\eta}{i}\left[J^{(i-1)}_{\frac{ie}{2}}-\omega^2 J^{(i+1)}_{\frac{ie}{2}}\right].\,i'\eta'\left[J^{(i'-1)}_{\frac{i'e'}{2}}-\omega'^2 J^{(i'+1)}_{\frac{i'e'}{2}}\right].$$

In the special case of $i=0$ the middle factors of these expressions take the indeterminate form $0\div0$, but then, in accordance with what has been shown above, we should read $-\frac{3}{2}e$. Thus, by means of the Besselian functions, these coefficients take finite forms.